LESSON

12 Related Rates

Once you have gotten the hang of implicit differentiation, it should not be difficult to take the derivative of both sides with respect to the variable t. This enables us to see how x and y vary with respect to time t. The only difference is that $\frac{d}{dt}(x) = \frac{dx}{dt}$, $\frac{d}{dt}(y) = \frac{dy}{dt}$, and so on. Only $\frac{d}{dt}(t) = 1$ can be simplified, but this generally never occurs.

Example

Differentiate $y^2 + \cos(x) = 4x^2y$ with respect to t.

Solution

Start with the equation.

$y^2 + \cos(x) = 4x^2y$

Differentiate both sides with respect to t.

$\frac{d}{dt}(y^2 + \cos(x)) = \frac{d}{dt}(4x^2y)$

Use the Chain Rule everywhere.

$$2y\cdot\frac{d}{dt}(y) - \sin(x)\cdot\frac{d}{dt}(x) =$$
$$8x\cdot\frac{d}{dt}(x)\cdot y + \frac{d}{dt}(y)\cdot 4x^2$$

Use $\frac{d}{dt}(x) = \frac{dx}{dt}$ and $\frac{d}{dt}(y) = \frac{dy}{dt}$.

$$2y\cdot\frac{dy}{dt} - \sin(x)\cdot\frac{dx}{dt} = 8xy\cdot\frac{dx}{dt} + \frac{dy}{dt}\cdot 4x^2$$

Example

Differentiate $e^x + y = y^3 + \sqrt{x}$ with respect to t.

Solution

Start with the equation.

$$e^x + y = y^3 + \sqrt{x}$$

Differentiate both sides with respect to t.

$$\frac{d}{dt}(e^x + y) = \frac{d}{dt}(y^3 + \sqrt{x})$$

Use the Chain Rule everywhere.

$$e^x\cdot\frac{d}{dt}(x) + \frac{d}{dt}(y) = 3y^2\cdot\frac{d}{dt}(y) + \frac{1}{2\sqrt{x}}\cdot\frac{d}{dt}(x)$$

Use $\frac{d}{dt}(x) = \frac{dx}{dt}$ and $\frac{d}{dt}(y) = \frac{dy}{dt}$.

$$e^x\cdot\frac{dx}{dt} + \frac{dy}{dt} = 3y^2\cdot\frac{dy}{dt} + \frac{1}{2\sqrt{x}}\cdot\frac{dx}{dt}$$

The variables need not be x and y.

Example

Differentiate $3A + 4B^2 = \frac{A}{r}$ with respect to t.

Solution

$$\frac{d}{dt}(3A + 4B^2) = \frac{d}{dt}\left(\frac{A}{r}\right)$$

$$3\cdot\frac{dA}{dt} + 8B\cdot\frac{dB}{dt} = \frac{\frac{dA}{dt}\cdot r - \frac{dr}{dt}\cdot A}{r^2}$$

Example

Differentiate $A = \pi r^2$ with respect to t.

Solution

$$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$$

$$\frac{dA}{dt} = 2\pi r\cdot\frac{dr}{dt}$$

Don't forget that π is a constant, not a variable!

▶ Practice

Differentiate with respect to t.

1. $y = (x^3 + x - 1)^5$
2. $y^4 - 3x^2 = \cos(y)$
3. $y^3 - y = 3x^4 - 10x^2 + 3x + 1$
4. $\sqrt{x} + \sqrt{y} = 10x^3 - 7x$
5. $\ln(y) + e^x = x^2y^2$
6. $5x^2 + 2x + 1 = w^2 + 7$
7. $z = \frac{2}{5}x^2 + \frac{2}{5}y^2 + \frac{3}{5x}$
8. $A^2 + B^2 = C^2$
9. $V = \frac{4}{3}\pi r^3$

10. $A = 4\pi r^2$

11. $C = 2\pi r$

12. $A = \frac{1}{2}bh$

Just as $\frac{dy}{dx} = \frac{y\text{-change}}{x\text{-change}}$ is a rate, so are $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dA}{dt}$, and so on. Because t usually represents time, $\frac{dy}{dt} = \frac{y\text{-change}}{t\text{-change}}$ represents how fast y is changing over time. Thus, if A is a variable that represents an area, $\frac{dA}{dt}$ represents how fast that area is increasing or decreasing.

Differentiating an equation with respect to t results in a new equation, which shows how the rates of change of the variables are related. For example, the area and radius of a circle are related by:

$$A = \pi r^2$$

If we differentiate with respect to t, we get:

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$

If a circle is growing in size, this equation details how the rate at which the radius is changing, $\frac{dr}{dt}$, relates to the rate at which the area is growing, $\frac{dA}{dt}$.

Example

A rock thrown into a pond makes a circular ripple that travels at 4 feet per second. How fast is the area of the circle increasing when the circle has a radius of 12 feet?

Solution

We know that for circles, $\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$. And we know that the radius is increasing at the rate of $\frac{dr}{dt} = 4$ feet per second, so when the radius is $r = 12$ feet, the area is increasing at:

$$\frac{dA}{dt} = 2\pi(12\text{ feet}) \cdot 4\frac{\text{feet}}{\text{second}}$$

$$= 96\pi\frac{\text{ft}^2}{\text{sec}}$$

$$= 96\pi \approx 301.6 \text{ square feet per second}$$

Example

A spherical balloon is inflated with 40 cubic inches of air every second. When the radius is 12 inches, how fast is the radius of the balloon increasing? (Hint: The volume of a sphere with radius r is $V = \frac{4}{3}\pi r^3$.)

Solution

We know that the volume of the balloon is increasing at the rate of $\frac{dV}{dt} = 40\frac{\text{in}^3}{\text{sec}}$. We want to know what $\frac{dr}{dt}$ is when $r = 12$ inches. If we differentiate $V = \frac{4}{3}\pi r^3$ with respect to t, we get:

$$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

When we plug in $\frac{dV}{dt} = 40\frac{\text{in}^3}{\text{sec}}$ and $r = 12$ in, we get:

$$40\frac{\text{in}^3}{\text{sec}} = 4\pi(12\,\text{in})^2 \cdot \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{40\text{ in}}{4\pi \cdot 144\text{ sec}} = \frac{5\text{in}}{72\pi\text{sec}}$$

The radius of the balloon is increasing at the very slow rate of $\frac{5}{72\pi} \approx 0.022$ inches per second.

Example

Suppose the base of a triangle is increasing at a rate of 8 feet per minute while the height is decreasing by 1 foot every minute. How fast is the triangle's area changing when the height is 5 feet and the base is 20 feet?

Solution

If we represent the length of the base by b, the height of the triangle as h, and the area of the triangle as A, then the formula that relates them all is $A = \frac{1}{2}bh$. The base is increasing at $\frac{db}{dt} = 8\frac{\text{ft}}{\text{min}}$ and the height is changing at $\frac{dh}{dt} = -1\frac{\text{ft}}{\text{min}}$. The -1 implies that 1 foot is subtracted from the height every minute, that is, the height is decreasing. We are trying to find $\frac{dA}{dt}$, which is the rate of change in area. When we differentiate our formula $A = \frac{1}{2}bh$ with respect to t, we get:

$$\frac{dA}{dt} = \frac{1}{2}\cdot\frac{db}{dt}\cdot h + \frac{dh}{dt}\cdot\frac{1}{2}b$$

When we plug in all of our information, including the $h = 5$ feet and $b = 20$ feet, we get:

$$\frac{dA}{dt} = \frac{1}{2}\cdot(8)\cdot(5) + (-1)\cdot\frac{1}{2}\cdot(20)$$

$$= 20 - 10 = 10$$

Thus, at the exact instant when the height is 5 feet and the base is 20, the area of the triangle is increasing at a rate of 10 square feet every minute.

Example

A 20 foot ladder slides down a wall at the rate of 2 feet per minute (see Figure 12.1). How fast is it sliding along the ground when the ladder is 16 feet up the wall?

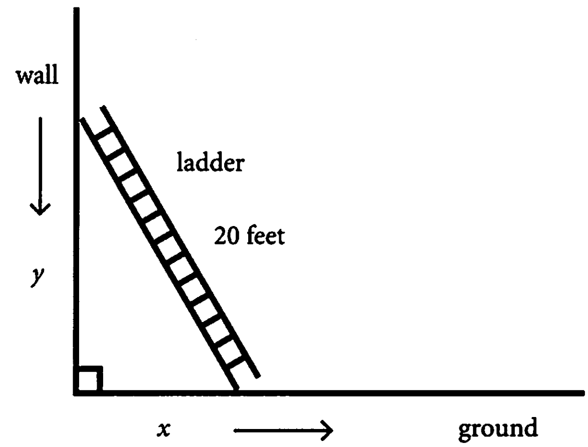


Figure 12.1

Solution

Here, $\frac{dy}{dt} = -2\frac{\text{ft}}{\text{min}}$ because the ladder is sliding down the wall at 2 feet per minute. We want to know $\frac{dx}{dt}$, the rate at which the bottom of the ladder is moving away from the wall. The equation to use is the Pythagorean theorem.

$$x^2 + y^2 = 20^2$$

$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(20^2)$$

$$2x\cdot\frac{dx}{dt} + 2y\cdot\frac{dy}{dt} = 0$$

If we plug in $y = 16$ and $\frac{dy}{dt} = -2$, we get:

$$2x\cdot\frac{dx}{dt} + 2(16)\cdot(-2) = 0$$

We still need to know what x is at the particular instant that $y = 16$, and for this, we go back to the Pythagorean theorem.

$$x^2 + (16)^2 = (20)^2$$

$$x^2 = 144, \text{ so } x = \pm 12$$

Changing Values Hint

It is important to use variables for all of the values that are changing. Only after differentiation can they be replaced by numbers.

Using $x = 12$ (a negative length here makes no sense), we get:

$$2 \cdot (12) \cdot \frac{dx}{dt} + 2 \cdot (16)(-2) = 0$$

$$\frac{dx}{dt} = \frac{8}{3}$$

At the moment that $y = 16$, the ladder is sliding along the ground at $\frac{8}{3}$ feet per minute.

In the previous example, it was okay to say that the hypotenuse was 20 because the length of the ladder didn't change. However, if we replace y with 16 in the equation before differentiating, we would have implied that the height was fixed at 16 feet. Because the height does change, it needs to be written as a variable, y. In general, anything that varies needs to be represented with a variable. Only after the derivative has been taken can the information for the given instant, like $y = 16$, be substituted.

▶ Practice

13. Suppose $y^2 + 3y = 6 - 4x^3$ and $\frac{dy}{dt} = 5$. What is $\frac{dx}{dt}$ when $x = -1$ and $y = 2$?

14. Suppose $xy^2 = x^2 + 3$. What is $\frac{dy}{dt}$ when $\frac{dx}{dt} = 8$, $x = 3$, and $y = -2$?

15. Let $K + e^L = L + I^2$. If $\frac{dL}{dt} = 5$ and $\frac{dI}{dt} = 4$, what is $\frac{dK}{dt}$ when $L = 0$ and $I = 3$?

16. Suppose $A^3 = B^2 + 4C^2$, $\frac{dA}{dt} = 8$, and $\frac{dC}{dt} = -2$. What is $\frac{dB}{dt}$ when $A = 2$, $B = 2$, and $C = 1$?

17. Suppose $A = I^2 + 6R$. If I increases by 4 feet per minute and R increases by 2 square feet every minute, how fast is A changing when $I = 20$?

18. Suppose $K^3 = \frac{1}{R^2} + 11$. Every hour, K increases by 2. How fast is R changing when $K = 3$ and $R = \frac{1}{4}$?

19. The height of a triangle increases by 2 feet every minute while its base shrinks by 6 feet every minute. How fast is the area changing when the height is 15 feet and the base is 20 feet?

20. The surface area of a sphere with radius r is $A = 4\pi r^2$. If the radius is decreasing by 2 inches every minute, how fast is the surface area shrinking when the radius is 20 inches?

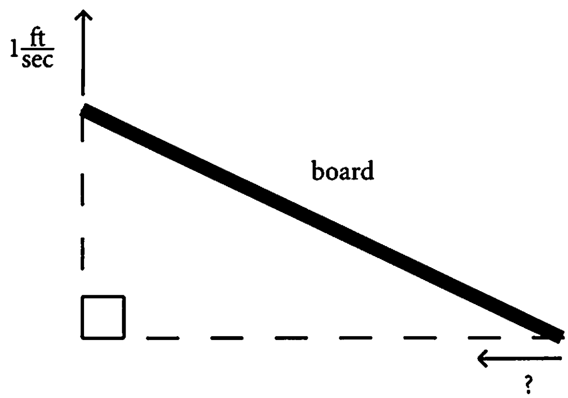
21. A circle increases in area by 20 square feet every hour. How fast is the radius increasing when the radius is 4 feet?

22. The volume of a cube grows by 1,200 square inches every minute. How fast is each side growing when each side is 10 inches?

23. The height of a triangle grows by 5 inches each hour. The area is increasing by 100 square inches

each hour. How fast is the base of the triangle increasing when the height is 20 inches and the base is 12 inches?

24. One end of a 10-foot long board is lifted straight off the ground at 1 foot per second (see figure below). How fast will the other end drag along the ground after 6 seconds?



25. A kite is 100 feet off the ground and moving horizontally at 13 feet per second (see figure below). How quickly must the string be let out when the string is 260 feet long?